Exercise 19

Solve the initial-value problem.

$$9y'' + 12y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$9(r^2 e^{rx}) + 12(re^{rx}) + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$9r^2 + 12r + 4 = 0$$

Solve for r.

$$(3r + 2)^2 = 0$$

$$r = \left\{-\frac{2}{3}\right\}$$

Two solutions to the ODE are $e^{-2x/3}$ and $xe^{-2x/3}$. By the principle of superposition, then,

$$y(x) = C_1 e^{-2x/3} + C_2 x e^{-2x/3}.$$

Differentiate the general solution.

$$y'(x) = -\frac{2}{3}C_1 e^{-2x/3} + C_2 e^{-2x/3} - \frac{2}{3}C_2 x e^{-2x/3}$$

Apply the initial conditions to determine C_1 and C_2.

$$y(0) = C_1 = 1$$

$$y'(0) = -\frac{2}{3}C_1 + C_2 = 0$$

Solving this system of equations yields $C_1 = 1$ and $C_2 = 2/3$. Therefore, the solution to the initial value problem is

$$y(x) = e^{-2x/3} + \frac{2}{3}xe^{-2x/3}.$$

Below is a graph of $y(x)$ versus x.